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An upper bound of the index of an equilibrium point in the plane

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ABSTRACT

We give an upper bound of the index of an isolated equilibrium point of a C^1 vector field in the plane. The vector field is decomposed in gradient and Hamiltonian components. This decomposition is related with the Loewner vector field. Associated to this decomposition we consider the set Π where the gradient and Hamiltonian components are linearly dependent. The number of branches of Π starting at the equilibrium point determines the upper bound of the index.

1. Introduction and the main results

The analysis of the trajectories near a non-hyperbolic isolated equilibrium point of a vector field in the plane is usually studied doing a blow-up of the point, but the method only works for analytic vector fields or more generally Lojasiewicz vector fields, see for instance [5]. A first approach to the dynamic near an equilibrium point is its topological index, for the definition see [5]. Here we give a procedure that can be applied to all C^1 vector fields on the plane. The method is based on the decomposition of the vector field as the difference of a gradient and a Hamiltonian vector field. The relation between the vector fields of this decomposition and the level sets of the Hamiltonian function give information on the structure of the trajectories near the equilibrium point and in particular an upper bound of the index.

Results on the estimation of the index of an equilibrium point are not abundant, see for instance [1–4,10,11] and the references quoted there. In many of these papers, as in our case, other vector fields or functions are used for the estimation: In [1–3] polynomial approximations, and in [10], for analytic systems, the scalar product of the position vector by the vector field.

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Our approach has been motivated by the Loewner Conjecture [9,7,16]. This conjecture generalizes the Caratheodory Conjecture that can be enunciated as follows: *Consider a C^r real-valued function f and the associated differential system*:

$$\begin{aligned}\frac{dx}{dt} &= v_1(x,y) = f_{xx} - f_{yy},\\ \frac{dy}{dt} &= v_2(x,y) = 2f_{xy}.\end{aligned} \tag{1}$$

Assume that this system has an isolated equilibrium point at the origin and $r \geqslant 3$. Then the index at the origin is less than or equal to two.

For an estimation of this index see [15].

The differential system (1) can be seen from a different point of view. Consider the Cauchy–Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then

$$\frac{\partial^2}{\partial \bar{z}^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2i\frac{\partial^2}{\partial x \partial y}\right).$$

The components of the vector field (1) can be identified with the real and imaginary part of the square of the Cauchy–Riemann operator.

A natural generalization of the differential system (1) is the vector field

$$\begin{aligned}L_n(f) &= \left(2^n\right)\left(\mathrm{Re}\frac{\partial^n}{\partial \bar{z}^n}, \mathrm{Im}\frac{\partial^n}{\partial \bar{z}^n}\right)(f)\\ &= \left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right)^n (f).\end{aligned}$$

If $n = 2$ we get the system (1). We can define also $L_0(f)$ as the vector field $(f, 0)$.

According to Titus [16] Loewner about 1950 conjectured: *If the vector field $L_n(f)$, $f \in C^\omega(D;\mathbb{R})$, has an isolated equilibrium point at the origin, then its index is not greater than n.* Note that the Loewner Conjecture coincides with the Caratheodory Conjecture when $n = 2$.

To state the main results we need to introduce some notations and definitions. Given a vector field $v(x,y)$, we will note by $\gamma_v(x,y)$ a trajectory of the vector field through the point (x,y); its flow will be denoted by φ_v, the index of an equilibrium point (x,y) by $i_v(x,y)$, and the foliation associated to the vector field on a domain D by $\mathcal{F}_u(D)$. We will denote by $\mathcal{E}_O$ a neighborhood, homeomorphic to an open disc centered at O. *From now on we will assume that the origin O of the plane is an isolated equilibrium point of the vector field $v(x,y)$ and all the functions and vector fields are defined in $\mathcal{E}_O$.*

For the definition of elliptic, hyperbolic and parabolic sector and sectorial decomposition of an equilibrium point see the references [5] or [8]. An equilibrium point is called a focus-center if there does not exist any separatrix arriving at it. All other equilibrium points will be called non-focus-center points. We recall the *Poincaré Index Formula* (see [5]). Given an isolated equilibrium point q with a finite sectorial decomposition, let e, h and p denote the number of elliptic, hyperbolic, and parabolic sectors of q, respectively. Then

$$i_v(q) = 1 + \frac{e-h}{2}.$$

The index is always an integer number.

The vector field $\nabla f = (f_x, f_y)$ is the *gradient vector field* associated to the function $f = f(x, y)$. The function f is strictly decreasing on the trajectories that are not equilibrium points. Therefore a gradient vector field cannot have elliptic sectors.

The *symplectic gradient* of a function $h = h(x, y)$ on a surface ∇_ω can be defined in local coordinates by

$$\nabla_\omega(h) = \frac{\partial h}{\partial y}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial y},$$

or simply $\nabla_\omega(h) = (h_y, -h_x)$, that is $\nabla_\omega(h)$ is the *Hamiltonian vector field* associated to the function h. The flow of a Hamiltonian vector field on the plane is area-preserving. Therefore in a sectorial decomposition of an equilibrium point of a Hamiltonian system cannot exist parabolic or elliptic sectors, and such equilibrium points cannot be foci. So all of them are centers or a collection of an even number of hyperbolic sectors.

By the Poincaré Index Formula the index of an isolated equilibrium point of a Hamiltonian or gradient vector field is always less than or equal to one.

Let $V^r(\mathbb{R}^2)$ be the space of C^r vector fields on the plane. For $r \geqslant 1$ we define the *Loewner map* $\Lambda : V^r(\mathbb{R}^2) \to V^{r-1}(\mathbb{R}^2)$ as

$$\Lambda\big((f, g)\big) = \nabla f - \nabla_\omega g.$$

This map generalizes the operator $L_n(f)$ in the following sense

$$L_{n+1}(f) = \Lambda\big(L_n(f)\big).$$

This formula follows easily by direct computations and in particular:

$$\Lambda\big(\Lambda(f, 0)\big) = (f_{xx} - f_{yy}, 2f_{xy}) = L_2(f).$$

The basic property of $\Lambda((f, g))$ is described in the next result.

Proposition 1. *Locally every C^1 vector field $v = (v_1, v_2)$ on the plane can be written as $v = \Lambda((f, g))$ for some functions f and g.*

This proposition is proved in [13] and generalized for the n-dimensional case in [12]. See also [14]. It is a consequence of the Hodge–De Rham theory, but it is easy to give a constructive method. In Section 2 we shall give an easy and constructive proof of it.

The Loewner Conjecture states that an upper estimation for the index of the equilibrium localized at the origin of a system $L_n(f)$ is n. Given an arbitrary vector field on the plane with an isolated equilibrium point, our initial question is: *What can be said about the index of this equilibrium point using the decomposition of the vector field given in Proposition* 1?

The main part of the paper is formed by Section 2. The initial point of our analysis is the dynamics of the Hamiltonian vector field of the decomposition. We will consider regions where the vector field is transversal to the foliation $\mathcal{F}_{\nabla_\omega(g)}(\mathcal{E}_O)$ of the Hamiltonian function.

More precisely we define the set Π as the points $(x, y) \in \mathcal{E}_O$ where $\nabla f(x, y)$ and $\nabla_\omega g(x, y)$ are linearly dependent.

On Π the vector fields $v(x, y), \nabla_\omega g(x, y)$ and $\nabla f(x, y)$ have the same direction. Analytically Π can be defined as the set of points (x, y) such that

$$P(x, y) = f_x g_x + f_y g_y = \nabla f \wedge \nabla g = 0. \tag{2}$$

Given a function $f : (\mathbb{R}^2, O) \to (\mathbb{R}, 0)$ with an isolated equilibrium point O, a *branch* is any one-dimensional connected component of $f^{-1}(\mathbb{R}) \setminus \{O\}$ restricted to a small disc centered at O. For more details on branches see [6].

Our two main results are the following.

Theorem 2. *Consider the vector field $v = \Lambda((f, g))$. Assume that the origin O is an equilibrium point of v and an isolated point of Π.*

(a) *If O is not an equilibrium point of $\nabla_\omega g$, then O as equilibrium point of v has two hyperbolic sectors and at most two parabolic sectors for the vector field v. Moreover $i_v(O) = 0$.*
(b) *If O is an equilibrium point of $\nabla_\omega g$, then all sectors of O as equilibrium point of v are parabolic or hyperbolic, and $i_v(O) \leqslant i_{\nabla_\omega g}(O)$.*

Theorem 3. *Consider the flow defined by $v = \Lambda((f, g))$. Assume that in a neighborhood of the origin Π consists of k branches starting at O, and that if O is an equilibrium point of $\nabla_\omega g$ it is isolated. Then the maximum number of elliptic sectors of O as equilibrium point of v is k, and*

$$i_v(O) \leqslant 1 + \frac{k}{2}. \tag{3}$$

The conditions of these two theorems cover almost all usual situations. What we do in all cases is a first approach to the dynamics near the equilibrium point, not only an estimation of the index. The key point is to prove that the maximum number of elliptic sectors in the decomposition of O is the number of branches of Π. By the Poincaré Index Formula we obtain an upper bound of $i_v(O)$. A lower bound can be obtained as follows. Consider a small disc $B_\varepsilon(O)$ that contains only one equilibrium point, namely O. This disc can be transformed to the sphere S^2 sending O and $\partial B_\varepsilon(O)$ to two equilibrium points S and N respectively. By the Poincaré–Hopf Theorem the index of S plus the index of N is equal to 2, therefore an upper estimation of the index of N gives a lower estimation of the index of S, i.e. of the index of O.

We finish the paper with some extensions and examples. If the set Π does not consist of k branches starting at O, but the points of Π outside the branches can be located in curves arriving at O, we define an extension of Π that we denote by Π^1. We get a new result generalizing Theorem 3. If we apply these theorems to Loewner Conjecture for $n = 2$ we get easily that generically this conjecture holds.

2. Dynamics of $\Lambda((f, g))$

Proof of Proposition 1. If $\nabla f - \nabla_\omega g = (v_1, v_2)$, then

$$f_x - g_y = v_1, \qquad f_y + g_x = v_2.$$

Therefore

$$f_{xy} - g_{yy} = v_{1_y}, \qquad f_{xy} + g_{xx} = v_{2_x}.$$

The function g is any solution of the Poisson equation

$$\Delta g = g_{xx} + g_{yy} = v_{2_x} - v_{1_y},$$

and f can be determined from $f_x = v_1 + g_y$. In fact $\Lambda((f, g)) = (v_1, v_2)$ implies $\Lambda((v_2, v_1)) = (\Delta g, \Delta f)$. $\square$

We denote by Z_f and Z_g the set of zeros of $\nabla f(x, y)$ and $\nabla_\omega g(x, y)$ respectively.

Given a function $h(x, y)$ we denote by $I_h(a)$ the level set of h corresponding to the value a, i.e. $I_h(a) = h^{-1}(a)$.

2.1. The set Π

Recall that the set Π consists of the points $(x, y) \in \mathcal{E}_O$ where $\nabla f(x, y)$ and $\nabla_\omega g(x, y)$ are linearly dependent.

We say that a curve is invariant by a vector field if it is formed by solution curves of this vector field. If a curve is invariant for $\nabla_\omega g(x, y)$ or $\nabla f(x, y)$ and also for the vector field $v(x, y)$, then the curve belongs to Π. Of course, the equilibrium point O of v is always a point of Π.

Proposition 4. *Given a neighborhood $\mathcal{E}_O$ of the equilibrium point of v localized at the origin O, on $\mathcal{E}_O \setminus \Pi$ the trajectories of $\gamma_v(x, y)$ intersect transversally $I_g(a)$ for all $a \in g(\mathcal{E}_O)$.*

Proof. To see that the trajectories $\gamma_v(x, y)$ intersect transversally $I_g(a)$ for all $a \in g(\mathcal{E}_O)$, it is sufficient to see that v is not parallel to $\nabla_\omega g$ in the points of $\mathcal{E}_O \setminus \Pi$. Since in $\mathcal{E}_O \setminus \Pi$ the vectors ∇f and $\nabla_\omega g$ are not parallel, $v = \nabla f - \nabla_\omega g$ is not parallel to $\nabla_\omega g$. □

In a similar way we define the set Q as the set of points (x, y) where $v(x, y)$ is tangent to $I_f(x, y)$. The conditions of tangency

$$v_1 = f_x - g_y = \lambda f_y, \qquad v_2 = f_y + g_x = -\lambda f_x$$

implies

$$f_x^2 + f_y^2 = f_x g_y - f_y g_x$$

or equivalently

$$|\nabla f|^2 = \nabla f \cdot \nabla_\omega g.$$

The structure of Π in a neighborhood of O can be very complicated. Three of the more easy possibilities are:

(i) O is an isolated point of Π.
(ii) O is not an isolated point of Π and there are finitely many branches of Π starting at O. In this case we call the point O a *ramification point* of Π. By construction Π consists of k branches that begin at O and $\mathcal{E}_O$ is divided in k sectors that we call *angular sectors* and denote by $S\Pi_i$ to avoid confusion with the sectors of the flow.
(iii) The condition (2), $P(x, y) = 0$, holds identically in $\mathcal{E}_O$. Then $v(x, y)$ can be considered as a Hamiltonian and a gradient system. The index of the equilibrium point at the origin is less than one. It is a trivial case and we do not consider again this possibility.

The number of branches through O can be obtained easily from any computer application that draws implicit plots and analytically by applying the following result of [6].

Proposition 5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a function germ with an isolated critical point 0. Let J_f be*

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ x & y \end{vmatrix}.$$

Assume that 0 *is also an isolated critical point of* J_f*. Then the number of branches of* $f^{-1}(0)$ *is* $2|\deg(f, J_f)|$*, where* $|\deg(f, J_f)|$ *is the absolute value of the topological degree of the mapping*

$$\frac{(f, J_f)}{\|(f, J_f)\|} : \mathbb{S}^1_\varepsilon \to \mathbb{S}^1_\varepsilon.$$

2.2. Basic properties

For proving our theorems we shall need some preliminary results.

Proposition 6. *Let* $v = \nabla f - \nabla_\omega g$*. An arc of the trajectory* $\gamma_{\nabla_\omega g}(x, y)$ *contained in the interior of an angular sector* $S\Pi_i$ *of* $\mathcal{E}_0 \setminus \Pi$ *cannot be intersected in two points by a connected piece of the orbit of* $\gamma_v(x, y)$ *completely contained in* $S\Pi_i$*.*

Proof. Assume that the proposition does not hold. Let $a = (x, y)$ be such that $\gamma_v(a)$ intersects $\gamma_{\nabla_\omega g}(a)$ in other points. Let b be the intersection point with the property that the arc $\widehat{ab} \subset \gamma_{\nabla_\omega g}(a)$ does not contain other intersection point. We can define a homeomorphism on the arc $\widehat{ab} \subset \gamma_{\nabla_\omega g}(a)$ into itself. Consider a point $p \in \widehat{ab}$ and $\gamma_v(p)$. Since the origin is an isolated equilibrium point of v, $\gamma_v(p)$ cannot remain in the compact region G defined by $\widehat{ab}$ and the arc of trajectory of $\gamma_v(a)$ from a to b. Therefore $\gamma_v(p)$, has another intersection point p_1 with $\gamma_{\nabla_\omega g}(a)$. This intersection point is unique since the transversality condition implies that $\gamma_v(p)$ do not enter again in G. The homeomorphism that sends p to p_1 reverses the orientation of the arc $\widehat{ab}$. Therefore it has a fixed point. At this fixed point, v and $\nabla_\omega g$ are parallel, in contradiction with the assumption. □

Proposition 7. *Let* p *be an equilibrium point of* $\nabla_\omega g$*,* Σ *a hyperbolic sector of* p *with respect to the vector field* $\nabla_\omega g$*. Assume that it is contained in the interior of an* $S\Pi$ *and let* s_1, s_2 *be the arcs of the separatrices of* Σ *contained in* $S\Pi$*. Assume that* s_1 *and* s_2 *are not invariant by* v*.*

(a) *For all* $(x, y) \in S\Pi_i \setminus \{s_1 \cup s_2\}$ *the trajectory of* $\gamma_v(x, y)$ *crosses* $s_1 \cup s_2$ *always from outside to the interior of* Σ*, or in the converse direction.*
(b) *An arc of* $\gamma_v(x, y)$ *contained in* $S\Pi_i$ *does not intersect* s_i *for* $i = 1, 2$*.*

Proof. To prove the first assertion, assume for instance that $\gamma_v(q)$, $q \in s_1$, cross s_1 from outside to inside Σ. Similar arguments can be used when $\gamma_v(q)$ cross s_1 from inside to outside.

If q_1 is close enough to q, by the continuity of v, $\gamma_v(q)$ crosses $\gamma_{\nabla_\omega g}(q_1)$. By the transversality conditions $\gamma_v(r)$, $r \in \gamma_{\nabla_\omega g}(q_1)$, crosses $\gamma_{\nabla_\omega g}(q_1)$ in the same sense that $\gamma_v(q_1)$ does. Since $\gamma_{\nabla_\omega g}(q_1)$ can be arbitrarily close to $s_1 \cup s_2$, the trajectories of v cross $s_1 \cup s_2$ in the same sense that $\gamma_{\nabla_\omega g}(q_1)$, therefore any γ_v can leave Σ.

To prove the second assertion, consider the same situation and notation of the last paragraph. Consider the region M defined by $\gamma_{\nabla_\omega g}(q_1)$, $s_1 \cup s_2$ contained in $S\Pi_i$. Then, $\gamma_v(q)$ leaves M. By Proposition 6 $\gamma_v(q)$ cannot enter M again. Therefore, $\gamma_v(q)$ do not intersect both s_i, $i = 1, 2$. □

Proposition 8. *Let* O *be an equilibrium point of* $\nabla_\omega g$*;* Σ_1 *and* Σ_2 *two adjacent hyperbolic sectors of the vector field* $\nabla_\omega g$ *contained in the same angular sector* $S\Pi_i$*,* s_1*,* s_2 *(resp.* s_2*,* s_3*) the arcs of the separatrices of* Σ_1 *(resp.* Σ_2*). If* s_2 *is not invariant for* $v(x, y)$ *there is a hyperbolic sector of* v *inside* $\Sigma_1 \cup \Sigma_2$ *such that* s_2 *is a section of the flow of* v*.*

Proof. The trajectories of v always cut s_2 in the same sense by Proposition 7. Since they cannot tend to the origin by Proposition 6 neither cut s_1 or s_3, they define a hyperbolic sector as stated in the proposition. □

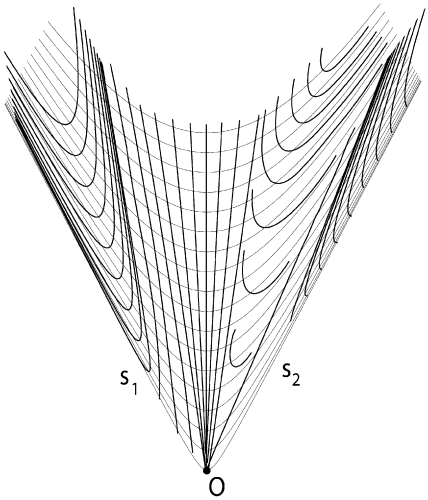

Fig. 1. Flow of v on a hyperbolic sector of $\nabla_\omega g$.

In fact inside a hyperbolic sector of $\nabla_\omega g(x, y)$ contained in the interior of an angular sector centered on O we can have parabolic or hyperbolic sectors of $v(x, y)$. In Fig. 1 we sketched such possibilities assuming that the separatrix s_1 is invariant for $v(x, y)$.

Proposition 9. *Consider the vector field $v(x, y)$. Assume that the origin O is an isolated equilibrium point of $\nabla_\omega g$. Then O has a finite sectorial decomposition for $\nabla_\omega g$.*

Proof. Assume that the proposition does not hold. Inside one of the k angular sectors of Π there are infinitely many hyperbolic sectors of $\nabla_\omega g$. By Proposition 8 the vector field $v(x, y)$ also has infinitely many hyperbolic sectors in the closure of $\mathcal{E}_O$. The sequence S_- of stable separatrices that has O as ω-limit point contains at least one limit separatrix s. But as the sequence S_+ of unstable separatrices that has O as α-limit point alternates with S_-, s is also a limit separatrix for S_+. By the orientability of the flow s is formed by equilibrium points of $v(x, y)$ in contradiction with the hypothesis that O is an isolated equilibrium point of $v(x, y)$. $\square$

2.3. Proofs of the two main theorems

We start proving Theorem 2, i.e. when O is an isolated point of Π.

Proof of Theorem 2. We can take $\mathcal{E}_O$ so that the set $\mathcal{E}_O \cap \Pi$ consists only of the equilibrium point O. Let $c = g(O)$.

If $O \notin Z_g$, then $\mathcal{F}_{\nabla_\omega g}(\mathcal{E}_O)$ is topologically equivalent to a foliation by straight lines. Let $\widehat{aOb}$ be an arc contained in $I_g(c) \cap \mathcal{E}_O$. Consider the union of the right open subarc $\widehat{aO}$ and the left open subarc $\widehat{Ob}$. As $v(x, y)$ is transversal to $\mathcal{F}_{\nabla_\omega g}(\mathcal{E}_O \setminus O)$, there is well defined a map between $\widehat{aO} \cup \widehat{Ob}$ and one arc, Υ_+ of $\mathcal{F}_{\nabla_\omega g}(\mathcal{E}_O)$ near to $\widehat{aOb}$ that sends a point q to $q_+ = \varphi_v(q, \varepsilon) = \gamma_v(q) \cap \Upsilon_+$ with the condition $\varphi_v(q, \varepsilon_1) \cap \Upsilon_+ = \emptyset$ if $0 < \varepsilon_1 < \varepsilon$. We assume that the end points of Υ_+ are the images of a and b. Let Υ'_+ be the image of this map. The closure of $\Upsilon_+ \setminus \Upsilon'_+$ are the initial points of trajectories that come from O. They define a parabolic sector or a separatrix. Considering now another map between $\widehat{aO} \cup \widehat{Ob}$ and one arc, Υ_- of $\mathcal{F}_{\nabla_\omega g}(\mathcal{E}_O)$ defined as the previous map but with negative time, we have another parabolic sector or separatrix. $\mathcal{E}_O$ minus the two parabolic sectors or separatrices consists of two hyperbolic sectors. Henceforth the index of O is zero and statement (a) is proved.

Assume now that $O \in Z_g$. If it is a center equilibrium point of $\nabla_\omega g$, it is surrounded by closed trajectories. As the vector field $v(x, y)$ is transversal to $\mathcal{F}_{\nabla_\omega g}(\mathcal{E}_O \setminus O)$ and only cuts a closed trajectory

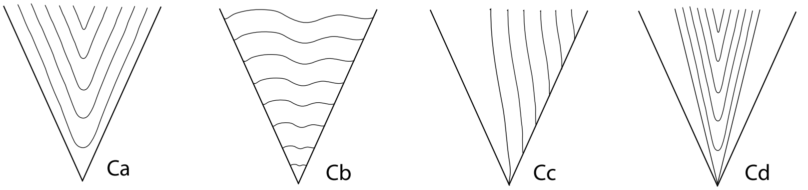

Fig. 2. $\mathcal{F}_{\nabla_\omega(g)}(S\Pi)$.

once, the function $g(x,y)-c$ is a strict Lyapunov function for a positive or negative sense of the time and O is an attractor or a repeller point. Therefore $i_v(O)=i_{\nabla_\omega g}(O)$.

If O is not a center equilibrium point of $\nabla_\omega g$, the local phase portrait of $\nabla_\omega g$ is formed by a set of hyperbolic sectors. In fact, by the definition of Π an equilibrium point of $\nabla_\omega g$ or ∇f belongs to Π. Therefore, if O is an isolated point of Π, O is also an isolated equilibrium point of $\nabla_\omega g(x,y)$ and Proposition 9 can be applied.

Proposition 8 implies that v at O has at least as many hyperbolic sectors as $\nabla_\omega g$. Therefore, by the Poincaré Index Formula statement (b) is proved in this case. □

If we consider the sets Q instead of the set Π, they can exist parabolic sectors in $\mathcal{F}_{\nabla_f}(\mathcal{E}_O)$ in a sectorial decomposition of O. But it is easy to see that even with these parabolic sectors we have estimations as in Theorem 2. In fact working in a similar way as we did for the set Π but now for the set Q we should have the following result.

Theorem 10. *Consider the vector field $v(x,y)$. Assume that the origin O is an isolated point of Q.*

(a) *If O is not an equilibrium point of ∇_f, then O has two hyperbolic sectors and at most two parabolic sectors for the vector field v. Moreover $i_v(O)=0$.*
(b) *If O is an equilibrium point of ∇_f, all sectors around O for the vector field v are parabolic or hyperbolic and $i_v(O)\leqslant i_{\nabla_f}(O)$.*

Now we shall study the case that the equilibrium point O is a ramification point of Π and that O is an isolated equilibrium point of $\nabla_\omega g$, then restricted to an angular sector $S\Pi_i$, the flow of $\nabla_\omega g$ is a Ca, Cb, or a finite combination of Cc and Cd regions represented in Fig. 2. To see that this covers all the possibilities we consider first that O is a center equilibrium point for $\nabla_\omega g$. Then $S\Pi_i$ is of type Cb. If O is not a center equilibrium point for $\nabla_\omega g$ it is surrounded by hyperbolic sectors. If one of these sectors coincides with an angular sector we are in case Ca. If $S\Pi_i$ is contained in a hyperbolic sector we are again in case Cb. If the angular and the hyperbolic sectors intersect and do not contain one to each other the type of region is Cc. The empty space in Cc of Fig. 2 will be filled by part of another hyperbolic sector. Finally if the angular sector contains a hyperbolic sector we are in case Cd. The last possibility is that $S\Pi_i$ contains several hyperbolic sectors or a subsector of a hyperbolic sector, then we have a finite combination of Cc and Cd.

Proof of Theorem 3. On each angular sector $S\Pi_i$ of type Ca, Cb the positive sense of the trajectories $\gamma_v(p)$ determines a total order on I_g. Then for any $p\in S\Pi_i$ the orbit $\gamma_v(p)\subset S\Pi_i$ cannot have simultaneously the α- and the ω-limit at O. Therefore there are no elliptic sectors at O with respect to the vector field v contained in $S\Pi_i$. Assume that $S\Pi_i$ is an angular sector of type Cc, Cd or a finite combination of Cc and Cd. Then we have a separatrix of $\nabla_\omega g$ inside the angular section, by Proposition 8 we have a hyperbolic sector of v or a subset of it inside the angular sector. All other orbits of v in $S\Pi_i$ define parabolic sectors (see Fig. 1) or leave the angular sector. Consequently the angular sector $S\Pi_i$ cannot contain a complete elliptic sector of v. One elliptic sector must be contained in at least two consecutive angular sectors, i.e. the number of elliptic sectors cannot exceed the number of branches of Π. Assuming this maximal possibility, by the Poincaré Index Formula we have the estimation of the theorem. □

3. An extension of the set Π

The set $P(x, y) = 0$ that defines Π has branches that begin at O and may be other points outside these branches. If these points do not accumulate to O they do not affect the dynamics of v near O. We are going to introduce, when it is possible, a new set Π^1 that contains the branches of Π and organize the other points of Π that accumulate to O inside new branches where not all the points are in Π.

We define the set Π^1 as the set of points in a neighborhood of O that verifies:

(i) $\Pi \subset \Pi^1$;
(ii) Π^1 minus the branches of Π consists also of k branches $B\Pi^1_j$ that begin at O;
(iii) each branch $B\Pi^1_j$ contains a sequence of points of Π that accumulates to O.

An angular sector for Π^1 is defined in the same way that it was defined for Π.

3.1. O is a ramification point of Π^1

The basic result of this section will be Theorem 11. In the hypothesis we do not require that O is an isolated equilibrium point of $\varphi_{\nabla_\omega g}$, therefore $i_{\nabla_\omega g}(O)$ can be not defined. Let $S\Pi^1$ be an angular sector, $B\Pi^1_1$ and $B\Pi^1_2$ the branches on its boundary, and $\partial S\Pi^1$ the intersection of $S\Pi^1$ with the border of $\mathcal{E}_O$. Consider a point on a $B\Pi^1_i$ a trajectory of $\nabla_\omega g$ can return to the same branch, intersect the other branch $\partial S\Pi^1$ or go to O.

Theorem 11. *Consider a $C^1(\mathbb{R}^2)$ vector field $v(x, y)$. Let the origin O be an isolated equilibrium point and $\Lambda((f, g))$ a decomposition of v. Assume that in a neighborhood of the origin Π^1 consists of k branches that begin at O. Then*

$$i_v(O) \leqslant 1 + \frac{k}{2}. \tag{4}$$

Proof. An angular sector $S\Pi^1$ of Π^1 whose limiting branches $B\Pi^1_j$, $j = 1, 2$, belong to Π, cannot contain a complete elliptic sector around O. If we prove the same property for angular sectors with one or two limiting branches contained in $\Pi^1 \setminus \Pi$ the formula (4) will be proved.

In all the proof $\{z_n\}$ will be a sequence of points converging to O on a branch $B\Pi^1_1$ or $B\Pi^1_2$.

Consider first that $\{z_n\} \in B\Pi^1_1$ is a sequence of points such that the trajectories of $\gamma_{\nabla_\omega g}(z_n)$ intersect also $B\Pi^1_2 \cup O$. By the transversality of v and $\nabla_\omega g$ on $S\Pi^1$ the orbits of v always cross one of these trajectories in the same sense. Given one trajectory, $\gamma_{\nabla_\omega g}(z_n)$ let q_n be the point of intersection with $B\Pi^1_2 \cup O$. The point q_n and O determine a closed arc on $B\Pi^1_2$. If there is a sequence of points $\{q_n\}$ whose limit is O as n tends to ∞ we have arcs $\widehat{z_n q_n}$ on any neighborhood of O. The orbits of v cross these arcs in one sense, therefore the angular sector cannot contain a complete elliptic sector. If there is not such sequence there is an arc $\widehat{Oq} \subset B\Pi^1_2$ without points q_n. Restrict $\mathcal{E}_O$ in such a way that any point q_n belongs to it. Then we do not have any $(z_n) \in B\Pi^1_1$ on the angular sector with the required intersection property. From now on we do not consider again the case of sequences $\{z_n\} \in B\Pi^1_1$ such that $\gamma_{\nabla_\omega g}(z_n)$ intersects $B\Pi^1_2 \cup O$.

Assume now that $\{\Gamma_n\} = \{\gamma_{\nabla_\omega g}(z_n)\} \cap S\Pi^1$ is a sequence of arcs of trajectories that intersect $\partial S\Pi^1$. On the closure of $S\Pi^1$ it must be a subsequence converging to an invariant curve $L_1 \subset S\Pi^1$ that goes from O to $\partial S\Pi^1$. We can assume that the initial sequence $\{\Gamma_n\}$ is the converging subsequence. The region limited by L_1, $\partial S\Pi^1$, Γ_n and the arc $\widehat{Oz_n} \subset BS\Pi^1_1$ will be called L_1R_n. These regions define a sequence of strips converging to L_1. L_1R_0 will be the region limited by L_1, $\partial S\Pi^1$, and $BS\Pi^1_1$. Consider a trajectory $\gamma_v(p)$ of $v(x, y)$ that intersects L_1. Since the intersection is transversal, $\gamma_v(p)$ crosses also an arc Γ_k near enough to L_1 and leaves, in a positive or negative sense, the region L_1R_k. By Proposition 6, from inside $L_1R_0 \setminus L_1R_k$ the trajectory $\gamma_v(p)$ cannot enter again into L_1R_k, therefore

$\gamma_v(p)$ cannot tend to O in L_1R_0. In the same way, a trajectory $\gamma_v(q)$ with q contained in the interior of L_1R_0 cannot have O as α and ω limit set since in this case $\gamma_v(q)$ will cut an arc Γ_n near enough to L_1.

To complete the description of the dynamics inside $S\Pi^1$ in this case we consider $BS\Pi_2^1$. There are three possibilities:

i) $BS\Pi_2^1$ is a branch of Π. For the Hamiltonian flow, the region $S\Pi^1 \setminus L_1R_0$ consists of a collection of hyperbolic sectors around O. By Proposition 8 and the remark, the trajectories of $v(x, y)$ that tend to O inside $S\Pi^1 \setminus L_1R_0$ define a parabolic sector or are separatrices. To have a trajectory of $v(x, y)$ homoclinic to O one of these separatrices need to come into L_1R_0 and tend to O in contradiction with the dynamics on L_1R_0.

ii) $BS\Pi_2^1$ is not a branch of Π and contains also a sequence of arcs of trajectories $\{\Gamma_n\}$ that intersect $\partial S\Pi^1$. We can repeat the construction made for $BS\Pi_1^1$ and find a region L_2R_0 similar to L_1R_0. We can repeat the arguments of i) considering now the region $S\Pi^1 \setminus (L_1R_0 \cup L_2R_0)$ instead of $S\Pi^1 \setminus L_1R_0$ to arrive at the impossibility of the existence of a trajectory of $v(x, y)$ homoclinic to O inside $S\Pi^1$. Observe that the region $S\Pi^1 \setminus (L_1R_0 \cup L_2R_0)$ can be empty if $L_1 = L_2$, in this case any orbit of $v(x, y)$ tends to O.

iii) $BS\Pi_2^1$ is not a branch of Π and do not contain a sequence of arcs of trajectories $\{\Gamma_n\}$ that intersect $\partial S\Pi^1$. Since we do not consider trajectories that cut the branches $BS\Pi_1^1$ and $BS\Pi_2^1$, the remaining possibility, taking $\mathcal{E}_O$ small enough, is that all trajectories through $BS\Pi_2^1$ cut again $BS\Pi_2^1$. Each trajectory $\phi = \gamma_{\nabla_\omega g}(p)$ defines a closed region ϕR limited by ϕ and $BS\Pi_2^1$. Consider the subset formed by the complete trajectories of the Hamiltonian vector field that pass through $BS\Pi_2^1$. It is by construction an invariant subset of $\nabla_\omega g$. The border is also an invariant subset. The intersection of this border with $S\Pi^1$ defines an invariant arc M. Since it is formed by solution curves of $\nabla_\omega g$ it must be C^1 except, may be, at the equilibrium points on it. Considering the region MR limited by M, $\partial S\Pi^1$, and $BS\Pi_2^1$, we can define an order relation on the trajectories $\gamma_{\nabla_\omega g}(p)$, $p \in MR$:

$$\phi_1 \prec \phi_2 \quad \text{if } \phi_1 R \supset \phi_2 R.$$

The least elements in this order are the trajectories that verify $\phi R \subset BS\Pi_2^1$.

Since the Hamiltonian vector field is transversal to the vector field $v(x, y)$ on MR, the trajectories of $v(x, y)$ cut the trajectories of $\nabla_\omega g$ in a way compatible with the order relation. Therefore all trajectories γ_v arrive at a least element of the order $\prec$ and leave MR through $BS\Pi_2^1$. The angular sector $S\Pi^1$ is formed by three subsets, MR, L_1R_0 and the complementary. Using the same arguments of ii) we conclude that in this case $S\Pi^1$ cannot contain an elliptic sector.

Assume finally that all trajectories through $BS\Pi_1^1$ cut again $BS\Pi_1^1$. As in case iii) we arrive at a region M_1R. If $BS\Pi_2^1$ is a branch of Π by arguments similar to i) we conclude that $S\Pi^1$ cannot contain an elliptic sector. The same conclusion follows if all trajectories through $BS\Pi_2^1$ cut again $BS\Pi_2^1$ by arguments similar to the case ii). $\square$

4. Examples

4.1. Example 1

Consider the vector field

$$v_1 = -y + x^3 - x^2y + 8xy^2,$$
$$v_2 = x - \frac{x^3}{3} + 8x^2y + y^3.$$

Then $v = \nabla f - \nabla_\omega g$ with

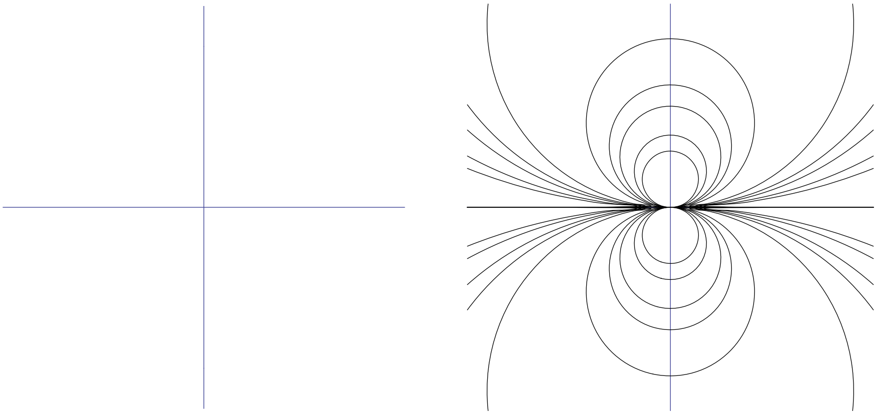

Fig. 3. Π and $\mathcal{F}_v(\mathcal{E}_O)$ for example 2.

$$f(x,y)=\frac{1}{12}\left(3x^4-4x^3y+48x^2y^2+3y^4\right),$$
$$g(x,y)=\frac{1}{2}\left(x^2+y^2\right).$$

The set Π is defined by:

$$P(x,y)=x^4-\frac{4}{3}x^3y+16x^2y^2+y^4=0.$$

The origin O is a minimum of $P(x,y)$ and therefore an isolated point of Π. The level sets of g are circles. We can apply the case where $O\in Z_g$ and is a center equilibrium point of $\nabla_\omega g$. Then the function $g(x,y)-g(O)$ is a strict Lyapunov function for a positive or negative sense of the time and O is an attractor or a repellor. Of course $i_v(O)=1$.

4.2. Example 2

Consider now the vector field:

$$\frac{dx}{dt}=x^2-y^2,$$
$$\frac{dy}{dt}=2xy.$$

By a direct inspection it is immediate to deduce that this vector field is the difference of the gradient of the function $\frac{x^3}{3}+xy^2$ and the symplectic gradient of $\frac{2}{3}y^3$. Computing the expression (2) we obtain that the set Π is defined by $4xy^3$. Therefore if has four branches; two of them corresponding to $y=0$ are invariant by φ_v. By formula (3) one gets $i_v(O)\leqslant 3$. See Fig. 3.

4.3. Example 3

The vector field:

$$v_1=-8x^3+7x^2y+21xy^2+5y^3,$$
$$v_2=x^3+30x^2y-xy^2-5y^3$$

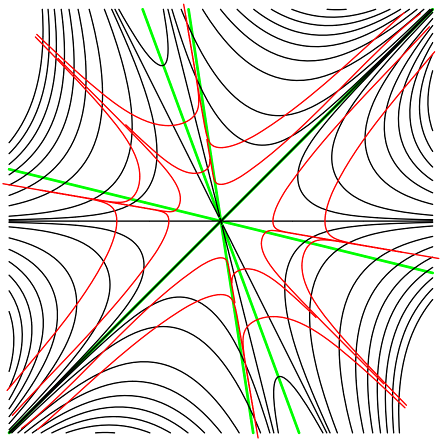

Fig. 4. $\mathcal{F}_{\nabla_\omega(g)}(\mathcal{E}_O)$, $\mathcal{F}_{\nabla(f)}(\mathcal{E}_O)$ and Π for example 3.

has the origin as an isolated zero. It can be written as the gradient of the function

$$f(x,y) = x^3y + 3x^2y^2 + xy^3,$$

minus the symplectic gradient of

$$g(x,y) = 8x^3y - 2x^2y^2 - 5xy^3 - y^4.$$

The set Π is defined by the condition:

$$P(x,y) = (x-y)\left(8x^5 + 52x^4y + 109x^3y^2 + 135x^2y^3 + 51xy^4 + 5y^5\right).$$

It is represented in Fig. 4.

The map of Proposition 5 (see Fig. 5) is:

$$\arctan\left(2\frac{-17\cos 2\theta - 180\cos 4\theta + 21\cos 6\theta + 84\sin 2\theta + 36\sin 4\theta - 96\sin 6\theta}{-6 + 84\cos 2\theta + 18\cos 4\theta - 32\cos 6\theta + 17\sin 2\theta + 90\sin 4\theta - 7\sin 6\theta}\right).$$

Since the degree of this map is four, by Theorem 3 an upper limit for the index of O for vector field $v(x,y)$ is five. The flow $\mathcal{F}_{\nabla_\omega(g)}(\mathcal{E}_O)$ is represented in Fig. 4.

The flow of $v(x,y)$ is represented in Fig. 6. The effective value of $i_v(O)$ is -3.

4.4. Example 4

Let now be

$$\begin{aligned} v_1 &= x^2 + y^2, \\ v_2 &= (x+y)^2 - x^4\cos\left(\frac{1}{x}\right) + 6x^5\sin\left(\frac{1}{x}\right), \\ (v_1, v_2) &= \nabla f - \nabla_\omega g, \\ f(x,y) &= \frac{(x+y)^3}{3} - \frac{y^3}{3}, \\ g(x,y) &= xy^2 + x^6\sin\left(\frac{1}{x}\right). \end{aligned}$$

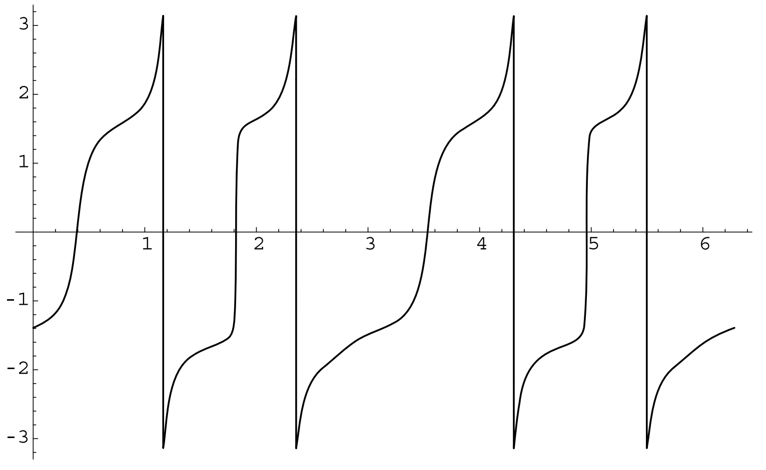

Fig. 5. Map of Proposition 5 for example 3.

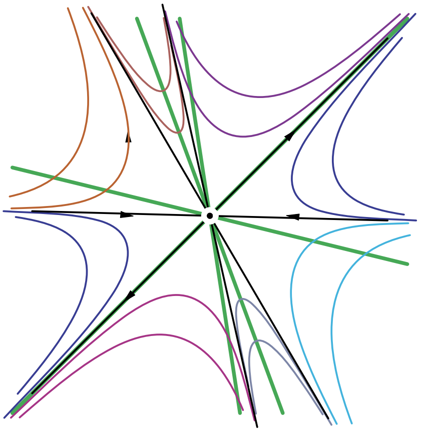

Fig. 6. Flow associated to v and the set Π for example 3.

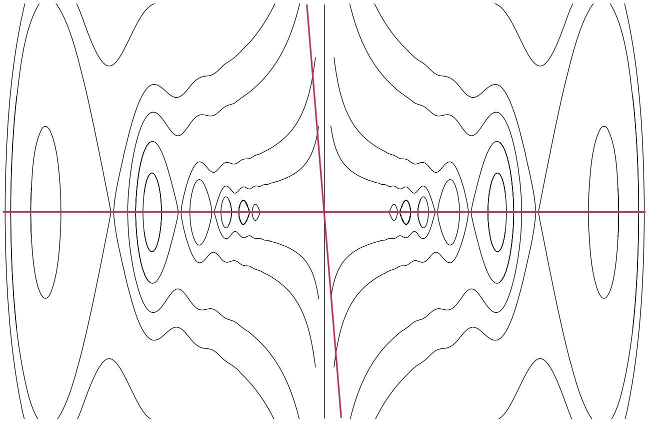

Fig. 7. $\mathcal{F}_{\nabla_\omega(g)}(\mathcal{E}_0)$ and Π^1 for example 4.

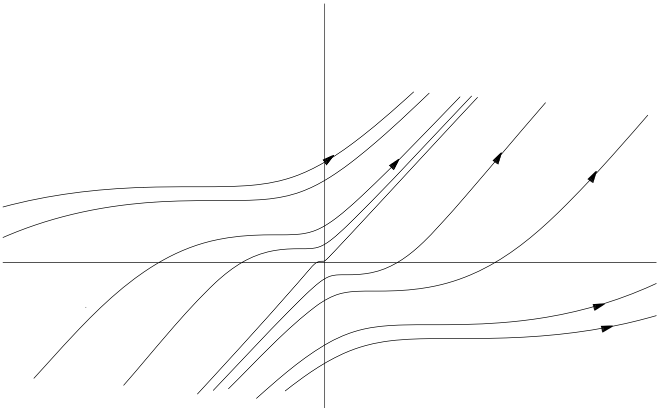

Fig. 8. The flow of example 4.

The set Π consists of one curve that lies in the second and fourth quadrants and the sequences of equilibrium points of the Hamiltonian system. This sequence lies on the OX axis. We extend the set Π to the set Π^1 adding the lines $y=0$. See Fig. 7. We have four angular sectors. In the sectors contained in the second and fourth quadrants there exists a sequence of points such that the trajectories of $\gamma_{\nabla_\omega g}(z_n)$ intersect also $B\Pi_2^1$. The other two angular sectors correspond to the item i) in the proof of Theorem 11.

The bound of $i_v(O)$ that Theorem 11 gives is three, but $i_v(O)=0$. In Fig. 8 it is represented φ_v, in $\mathcal{E}_O$.

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References

[1] V.I. Arnold, Index of a singular point of a vector field, the Petrovski–Oleini inequality, and mixed Hodge structures, Funct. Anal. Appl. 2 (1978) 1–11.
[2] F.S. Berezovskaya, Index of a stationary point of a vector field on the plane, Funct. Anal. Appl. 13 (2) (1979) 137.
[3] M. Brunella, M. Miari, Topological equivalence of a plane vector field with its principal part defined through Newton polyhedra, J. Differential Equations 85 (1990) 338–366.
[4] A. Cima, A. Gasull, J. Torregrosa, On the relation between index and multiplicity, J. London Math. Soc. 57 (1998) 757–768.
[5] F. Dumortier, J. Llibre, J.C. Artés, Qualitative Theory of Planar Differential Systems, Universitext, Springer, 2006.
[6] T. Fukuda, K. Aoki, W. Sun, On the number of branches of a plane curve germ, Kodai Math. J. 9 (1986) 179–187.
[7] B. Guilfoyle, W. Klingenberg, Proof of the conjecture of Carathéodory conjecture by mean curvature flow in the space of oriented affine lines, arXiv:0808.0851v1, 6 August 2008.
[8] P. Hartman, Ordinary Differential Equations, John Wiley and Sons, Inc., 1964.
[9] V.V. Ivanov, The analytic Carathéodory conjecture, Sib. Math. J. 43 (2002) 251–322.
[10] M. Izydorek, S. Rybicki, Z. Szafraniec, A note on the Poincaré–Bendixson index theorem, Kodai Math. J. 19 (1996) 145–156.
[11] J. Llibre, R. Saghin, The index of singularities of vector fields and finite jets, J. Differential Equations 251 (2011) 2822–2832.
[12] R. Mendes, J. Duarte, Decomposition of vector fields and mixed dynamics, J. Math. Phys. 22 (1981) 1420–1422.
[13] J. Roels, On the local decomposition of a vector field on a symplectic surface as the sum of a gradient and a Hamiltonian vector field, C. R. Acad. Sci. Paris. Ser. A 78 (1974) 29–31.
[14] M.T. Scholes, Vector field decomposition and first integrals with applications to non-linear systems, Thesis, University of Johannesburg, 2006.
[15] B. Smyth, F. Xavier, Eigenvalue estimates and the index of Hessian fields, Bull. Lond. Math. Soc. 33 (2001) 109–112.
[16] C.J. Titus, A proof of a conjecture of Loewner and of the conjecture of Carathéodory on umbilic points, Acta Math. 131 (1973) 43–77.